A semi-classical trace formula at a totally degenerate critical level.
Contributions of local extremum.

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Abstract

We study the semi-classical trace formula at a critical energy level for an h-pseudo-differential operator on $\mathbb{R}^n$ whose principal symbol has a totally degenerate critical point for that energy. This problem is studied for a large time behavior and under the hypothesis that the principal symbol of the operator has a local extremum at the critical point.

1 Introduction.

The semi-classical trace formula for a self-adjoint h-pseudo-differential operator P_h, or more generally h-admissible (see [17]), studies the asymptotic behavior, as h goes to 0, of the spectral function:

$$\gamma(E, h, \varphi) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \varphi(\frac{\lambda_j(h) - E}{h}), \tag{1}$$

where the $\lambda_j(h)$ are the eigenvalues of P_h and where we suppose that the spectrum is discrete in $[E - \varepsilon, E + \varepsilon]$, some sufficient conditions for this are given below. If p_0 is the principal symbol of P_h we recall that an energy E is regular when $\nabla p_0(x, \xi) \neq 0$ on the energy surface $\Sigma_E = \{(x, \xi) \in T^\star\mathbb{R}^n \ / \ p_0(x, \xi) = E\}$ and critical when it is not regular.

A classical property is the existence of a link between the asymptotics of (1), as h tends to 0, and the closed trajectories of the Hamiltonian flow of p_0 on Σ_E

$$\lim_{h \to 0} \gamma(E, h, \varphi) \rightleftharpoons \{(t, x, \xi) \in \mathbb{R} \times \Sigma_E \ / \ \Phi_t(x, \xi) = (x, \xi)\},$$

where $\Phi_t = \exp(tH_{p_0})$ and $H_{p_0} = \partial_\xi p_0.\partial_x - \partial_x p_0.\partial_\xi$. A non-exhaustive list of references concerning this subject is Gutzwiller [10], Balian and Bloch [1] for the physic literature and for a mathematical point of view Brummelhuis and Uribe

[3], Paul and Uribe [15], and more recently Combescure, Ralston and Robert [7], Petkov and Popov [16], Charbonnel and Popov [6].

The case of a non-degenerate critical energy of the principal symbol $p_0(x, \xi)$, that is such that the critical-set $\mathfrak{C}(p_0) = \{(x, \xi) \in T^*\mathbb{R}^n \ / \ dp_0(x, \xi) = 0\}$ is a compact C^∞ manifold with a Hessian $d^2 p_0$ transversely non-degenerate along this manifold, has been investigated first by Brummelhuis, Paul and Uribe in [2]. They treated this question for quite general operators but for some "small times", that is it was assumed that 0 is the only period of the linearized flow in $\mathrm{supp}(\hat{\varphi})$ when it is small. Later, Khuat-Duy in [13] and [14] has obtained the contributions of the non-zero periods of the linearized flow with the assumption that $\mathrm{supp}(\hat{\varphi})$ is compact, but for Schrödinger operators with symbol $\xi^2 + V(x)$ and a non-degenerate potential V. Our contribution to this subject was to compute the contributions of the non-zero periods of the linearized flow for some more general operators, always with $\hat{\varphi}$ of compact support and under some geometrical assumptions on the flow (see [4] or [5]).

Basically, the asymptotics of (1) can be expressed in terms of oscillatory integrals whose phases are related to the classical dynamics of p_0 on Σ_E. For (x_0, ξ_0) a critical point of p_0, it is well known that the relation:

$$\mathrm{Ker}(d_{x,\xi}\Phi_t(x_0, \xi_0) - \mathrm{Id}) \neq \{0\}, \tag{2}$$

leads to the study of degenerate oscillatory integrals. What is new here is that we examine the case of a totally degenerate energy, that is such that the Hessian matrix at our critical point is zero. Hence, the linearized flow for such a critical point satisfies $d_{x,\xi}\Phi_t(x_0, \xi_0) = \mathrm{Id}$, for all $t \in \mathbb{R}$ and the oscillatory integrals we have to consider are totally degenerate.

The results obtained here are global in time, that is we only assume that $\mathrm{supp}(\hat{\varphi})$ is compact. The core of the proof lies in establishing suitable normal forms for our phase functions and in a generalization of the stationary phase formula for these normal forms.

2   Hypotheses and main result.

Let $P_h = Op_h^w(p(x, \xi, h))$ be a h-pseudodifferential operator in the class of the h-admissible operators with symbol $p(x, \xi, h) \sim \sum h^j p_j(x, \xi)$, i.e. there exist sequences $(p_j)_j \in \Sigma_0^m(T^*\mathbb{R}^n)$ and $(R_N(h))_N$ such that:

$$P_h = \sum_{j<N} h^j p_j^w(x, hD_x) + h^N R_N(h), \ \forall N \in \mathbb{N},$$

where $R_N(h)$ is a bounded family of operators on $L^2(\mathbb{R}^n)$, for $h \leq h_0$, and:

$$\Sigma_0^m(T^*\mathbb{R}^n) = \{a : T^*\mathbb{R}^n \to \mathbb{C}, \ \sup |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < C_{\alpha, \beta} m(x, \xi), \ \forall \alpha, \beta \in \mathbb{N}^n\},$$

where m is a tempered weight on $T^*\mathbb{R}^n$. For a detailed exposition on h-admissible operators we refer to the book of Robert [17]. Let us note $p_0(x, \xi)$ the

principal symbol of P_h, $p_1(x,\xi)$ the sub-principal symbol and $\Phi_t = \exp(tH_{p_0})$: $T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, the Hamiltonian flow of $H_{p_0} = \partial_\xi p_0.\partial_x - \partial_x p_0.\partial_\xi$.

If E_c is a critical value of p_0, we study the asymptotics of the spectral distribution

$$\gamma(E_c, h) = \sum_{\lambda_j(h) \in [E_c - \varepsilon, E_c + \varepsilon]} \varphi(\frac{\lambda_j(h) - E_c}{h}), \qquad (3)$$

under the hypotheses (H_1) to (H_4) given below.

(H_1) *The symbol of P_h is real.*
There exists $\varepsilon_0 > 0$ such that $p_0^{-1}([E_c - \varepsilon_0, E_c - \varepsilon_0])$ is compact.

Then, by Theorem 3.13 of [17] the spectrum $\sigma(P_h) \cap [E_c - \varepsilon, E_c + \varepsilon]$ is discrete and consists in a sequence $\lambda_1(h) \leq \lambda_2(h) \leq ... \leq \lambda_j(h)$ of eigenvalues of finite multiplicities, if $\varepsilon < \varepsilon_0$ and h is small enough. To simplify notations we write $z = (x, \xi)$ for any point of the phase space.

(H_2) *On the energy surface $\Sigma_{E_C} = p_0^{-1}(\{E_c\})$, p_0 has a unique critical point $z_0 = (x_0, \xi_0)$ and near z_0 :*

$$p_0(z) = E_c + \sum_{j=k}^{N} \mathfrak{p}_j(z) + \mathcal{O}(||(z - z_0)||^{N+1}), \ k > 2,$$

where the functions $\mathfrak{p}_j$ are homogeneous of degree j in $z - z_0$.
(H_3) *We have $\hat{\varphi} \in C_0^\infty(\mathbb{R})$.*

With (H_2) the oscillatory-integrals we will have to consider are totally degenerate. Hence, they cannot be treated by the classical stationary phase method. To solve this problem we impose the following condition on the symbol:

(H_4) *The critical point z_0 is a local extremum of p_0.*

Remark 1 (H_4) implies that the first non-zero homogeneous component $\mathfrak{p}_k$ is even and is positive or negative definite and also that z_0 is isolated on Σ_{E_c}.

Since we are interested in the contribution to the trace formula of the fixed point z_0, to understand the new phenomenon, it suffices to study:

$$\gamma_{z_0}(E_c, h) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t) \psi^w(x, hD_x) \exp(-\frac{i}{h} tP_h) \Theta(P_h) dt. \qquad (4)$$

Here Θ is a function of localization near the critical energy surface Σ_{E_c} and $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ has an appropriate support near z_0. Rigorous justifications are given in section 3 for the introduction of $\Theta(P_h)$ and in section 4 for $\psi^w(x, hD_x)$. Then, the new contribution to the trace formula is given by:

Theorem 2 *Under hypotheses (H_1) to (H_4) we obtain:*

$$\gamma_{z_0}(E_c, h) \sim h^{\frac{2n}{k} - n}(\sum_{j=0}^{N} \Lambda_{j,k}(\varphi) h^{\frac{j}{k}} + \mathcal{O}(h^{\frac{N+1}{k}})), \ as \ h \to 0,$$

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where the $\Lambda_{j,k}$ are some distributions and the leading coefficient is given by:

$$\Lambda_{0,k}(\varphi) = \frac{1}{k} \left\langle \varphi(t + p_1(z_0)), t_{z_0}^{\frac{2n-k}{k}} \right\rangle \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{2n-1}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta, \qquad (5)$$

with $t_{z_0} = \max(t, 0)$ if z_0 is a minimum and $t_{z_0} = \max(-t, 0)$ for a maximum.

Remark 3 One can derive a full asymptotic expansion as shows Lemma 16.

3 Oscillatory representation.

Let be $\varphi \in \mathcal{S}(\mathbb{R})$ with $\hat{\varphi} \in C_0^\infty(\mathbb{R})$. We recall that:

$$\gamma(E_c, h) = \sum_{\lambda_j(h) \in I_\varepsilon} \varphi(\frac{\lambda_j(h) - E_c}{h}), \ I_\varepsilon = [E_c - \varepsilon, E_c + \varepsilon],$$

with $p_0^{-1}(I_{\varepsilon_0})$ compact in $T^*\mathbb{R}^n$ the spectrum of P_h is discrete in I_ε for $\varepsilon < \varepsilon_0$ and h small enough. Now, we localize near the critical energy E_c with a cut-off function $\Theta \in C_0^\infty(]E_c - \varepsilon, E_c + \varepsilon[)$, such that $\Theta = 1$ near E_c and $0 \le \Theta \le 1$ on $\mathbb{R}$. The associated decomposition is:

$$\gamma(E_c, h) = \gamma_1(E_c, h) + \gamma_2(E_c, h), \qquad (6)$$

with:

$$\gamma_1(E_c, h) = \sum_{\lambda_j(h) \in I_\varepsilon} (1 - \Theta)(\lambda_j(h))\varphi(\frac{\lambda_j(h) - E_c}{h}),$$

$$\gamma_2(E_c, h) = \sum_{\lambda_j(h) \in I_\varepsilon} \Theta(\lambda_j(h))\varphi(\frac{\lambda_j(h) - E_c}{h}).$$

The asymptotic behavior of $\gamma_1(E_c, h)$ is given by:

Lemma 4 $\gamma_1(E_c, h) = \mathcal{O}(h^\infty)$ as $h \to 0$.

Proof. Since $\varphi \in \mathcal{S}(\mathbb{R})$, $\forall k \in \mathbb{N}$, $\exists C_k$ such that $|x^k \varphi(x)| \le C_k$ on $\mathbb{R}$. By Theorem 3.13 of [17] the number of eigenvalues $N(h)$ lying in $I_\varepsilon \cap \mathrm{supp}(1 - \Theta)$ is of order $\mathcal{O}(h^{-n})$, for h small enough. This gives the estimate:

$$|\gamma_1(E_c, h)| \le N(h)C_k |\frac{\lambda_j(h) - E_c}{h}|^{-k}.$$

But on the support of $(1 - \Theta)$ we have $|\lambda_j(h) - E_c| > \varepsilon_0 > 0$. This leads to:

$$|\gamma_1(E_c, h)| \le N(h)C_N \varepsilon_0^{-k} h^k \le c_N h^{k-n}.$$

Since the property is true for all $k \in \mathbb{N}$, this ends the proof. ∎

Consequently, for the study of $\gamma(E_c, h)$ modulo $\mathcal{O}(h^\infty)$, we have only to consider the quantity $\gamma_2(E_c, h)$. By inversion of the Fourier transform we have:

$$\Theta(P_h)\varphi(\frac{P_h - E_c}{h}) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t)\exp(-\frac{i}{h}tP_h)\Theta(P_h)dt.$$

Since the trace of the left hand-side is exactly $\gamma_2(E_c, h)$, we obtain:

$$\gamma_2(E_c, h) = \frac{1}{2\pi}\text{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t)\exp(-\frac{i}{h}tP_h)\Theta(P_h)dt, \tag{7}$$

and with Lemma 4 this gives:

$$\gamma(E_c, h) = \frac{1}{2\pi}\text{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t)\exp(-\frac{i}{h}tP_h)\Theta(P_h)dt + \mathcal{O}(h^\infty).$$

Remark 5 Another interest of this formulation is that, under the geometrical condition to have a "clean" flow, $\gamma(E_c, h)$ can be expressed, at the first order, as the composition of two Fourier integral-operators.

Let be $U_h(t) = \exp(-\frac{it}{h}P_h)$ the evolution operator. For each integer L we can approximate $U_h(t)\Theta(P_h)$, modulo $\mathcal{O}(h^L)$, by a Fourier integral-operator, or FIO, depending on a parameter h. To give a precise formulation of this approximation we recall briefly the principal notions on FIO. Let be $N \in \mathbb{N}$, $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ and $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. An oscillatory-integral with phase φ and amplitude a is

$$I(ae^{\frac{i}{h}\varphi}) = (2\pi h)^{-\frac{N}{2}} \int_{\mathbb{R}^N} a(x, \theta)e^{\frac{i}{h}\varphi(x,\theta)}d\theta. \tag{8}$$

To attain our objectives, it suffices to consider amplitudes a with compact support. We suppose, as usually, that $\varphi = \varphi(x, \theta)$ is a non-degenerate phase function, i.e.

$$d(\partial_{\theta_1}\varphi) \wedge ... \wedge d(\partial_{\theta_N}\varphi) \neq 0 \text{ on } \mathfrak{C}(\varphi) = \{(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^N \ / \ d_\theta\varphi(x, \theta) = 0\}.$$

This implies that $\mathfrak{C}(\varphi)$ is a sub-manifold of class C^∞ of $\mathbb{R}^n \times \mathbb{R}^N$ and that:

$$i_\varphi : \begin{cases} \mathfrak{C}(\varphi) \to T^*(\mathbb{R}^n), \\ (x, \theta) \mapsto (x, d_x\varphi(x, \theta)), \end{cases}$$

is an immersion. In this situation one say that φ parameterizes the Lagrangian manifold $\Lambda_\varphi = i_\varphi(\mathfrak{C}(\varphi))$. Conversely, if $\Lambda \subset T^*\mathbb{R}^n$ is a Lagrangian sub-manifold we can always find locally some non-degenerate phase functions parameterizing Λ, see e.g. [8]. Now if φ_1 and φ_2, $\varphi_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{N_j})$, are two non-degenerate phase functions parameterizing locally the same Lagrangian manifold, i.e. $\Lambda_{\varphi_1} \cap$

$U = \Lambda_{\varphi_2} \cap U$, for an open $U \subset T^*\mathbb{R}^n$, then it is well-known that there exists a constant $c \in \mathbb{R}$ such that for all $a_1 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^{N_1})$ with $i_{\varphi_1}(\operatorname{supp}(a_1) \cap \mathfrak{C}(\varphi_1)) \subset U$, small enough, there exists a sequence $a_{2,j} \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^{N_2})$, $j \in \mathbb{N}$, such that for all $L \in \mathbb{N}$:

$$I(a_1 e^{\frac{i}{\hbar}\varphi_1}) = e^{\frac{i}{\hbar}c} \sum_{j < L} h^j I(a_{2,j} e^{\frac{i}{\hbar}\varphi_2}) + h^L r_L(h),$$

with $r_L(h)$ uniformly bounded in $L^2(\mathbb{R}^n)$, since $a_1 \in C_0^\infty$, and c comes from $S_{\varphi_1} - S_{\varphi_2} = c$ on $\Lambda_{\varphi_1} \cap U = \Lambda_{\varphi_2} \cap U$. This recalls that the fundamental object associated to an oscillatory-integral is the Lagrangian manifold parameterized by the phase function. The next definition follows Hörmander.

Definition 6 *Let be $\Lambda \subset T^*\mathbb{R}^n$ a Lagrangian sub-manifold of class C^∞. The class $I(\mathbb{R}^n, \Lambda)$ of oscillatory functions associated to Λ is given by:*

$$u_h \in I(\mathbb{R}^n, \Lambda) \Leftrightarrow u_h = \sum_v I(a_v(h) e^{\frac{i}{\hbar}\varphi_v}),$$

where the sum is locally finite and the functions $a_v(h)$ are of the form:

$$a_v(h) = \sum_{-d_v \le j < J_v} h^j a_j, \ a_j \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{N_v}).$$

With d_v, J_v and N_v in $\mathbb{N}$ and where φ_v is a non degenerate phase function parameterizing $\Lambda \cap i_{\varphi_v}(\operatorname{supp}(a_v(h)))$, where $\operatorname{supp}(a_v(h)) = \bigcup_j \operatorname{supp}(a_{v,j})$.

Now let be $\Lambda \subset T^*(\mathbb{R}^n \times \mathbb{R}^l)$ a C^∞ Lagrangian sub-manifold of $T^*(\mathbb{R}^n \times \mathbb{R}^l)$.

Definition 7 *The family of operators $(F_h) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^l)$, $0 < h \le 1$, is a h-FIO, associated to Λ, if there exists two families $\hat{F}_h^{(N)}$ and $R_h^{(N)}$, $N \in \mathbb{N}$, such that:*
i) Each $\hat{F}_h^{(N)}$ has an integral kernel in $I(\mathbb{R}^n \times \mathbb{R}^l, \Lambda)$.
ii) For each N the operator $R_h^{(N)}$ is bounded on L^2 and there exists $C_N > 0$ such that $\|R_h^{(N)}\|_{\mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^l))} \le C_N$, uniformly in h, $0 < h \le 1$.
iii) For each $N : F_h = \hat{F}_h^{(N)} + h^N R_h^{(N)}$.

After these definitions, we recall the theorem that gives the approximation of $U_h(t)\Theta(P_h)$. Let Λ be the Lagrangian manifold associated to the flow of p_0:

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \tau = p_0(x, \xi), \ (x, \xi) = \Phi_t(y, \eta)\}.$$

Theorem 8 *The operator $U_h(t)\Theta(P_h)$ is an h-FIO associated to Λ, there exists $U_{\Theta,h}^{(N)}(t)$ with integral kernel in $I(\mathbb{R}^{2n+1}, \Lambda)$ and $R_h^{(N)}(t)$ bounded, with a L^2-norm uniformly bounded for $0 < h \le 1$ and t in a compact subset of $\mathbb{R}$, such that $U_h(t)\Theta(P_h) = U_{\Theta,h}^{(N)}(t) + h^N R_h^{(N)}(t)$.*

We refer to Duistermaat [8] for a proof of this theorem.

Remark 9 By a theorem of Helffer and Robert, see e.g. [17], Theorem 3.11 and Remark 3.14, $\Theta(P_h)$ is an h-admissible operator with a symbol of compact support in $p_0^{-1}(I_\varepsilon)$. This allows us to consider only oscillatory-integrals with compact support.

For our goal the following corollary is crucial.

Corollary 10 *Let be* $\Theta_1 \in C_0^\infty(\mathbb{R})$ *such that* $\Theta_1 = 1$ *on* $\operatorname{supp}(\Theta)$ *and* $\operatorname{supp}(\Theta_1) \subset I_\varepsilon$, *then* $\forall N \in \mathbb{N}$:

$$\operatorname{Tr}(\Theta(P_h)\varphi(\frac{P_h - E_c}{h})) = \frac{1}{2\pi}\operatorname{Tr}\int_{\mathbb{R}} \hat{\varphi}(t)e^{\frac{i}{h}tE_c}U_{\Theta,h}^{(N)}(t)\Theta_1(P_h)dt + \mathcal{O}(h^N).$$

Proof. By cyclicity of the trace, for all $N \in \mathbb{N}$ we have:

$$\operatorname{Tr}(\Theta(P_h)\varphi(\frac{P_h - E_c}{h})) = \frac{1}{2\pi}\operatorname{Tr}\int_{\mathbb{R}} \hat{\varphi}(t)e^{\frac{i}{h}tE_c}(U_{\Theta,h}^{(N)}(t) + h^N R_h^{(N)}(t))\Theta_1(P_h)dt.$$

But $\Theta_1(P_h)$, and a fortiori $\Theta_1(P_h)R_h^{(N)}(t)$ by the ideal property, is a class-trace operator (see e.g. [17], section 2.5), with norm:

$$||\Theta_1(P_h)R_h^{(N)}(t)||_{\operatorname{Tr}} \leq ||R_h^{(N)}(t)|| \; ||\Theta_1(P_h)||_{\operatorname{Tr}} \leq C_{\hat{\varphi}}||\Theta_1(P_h)||_{\operatorname{Tr}},$$

since it is true for all $t \in \operatorname{supp}(\hat{\varphi})$ the corollary is proven. $\blacksquare$

The geometrical approach associated to the Hörmander class $I(\mathbb{R}^{2n+1}, \Lambda)$ gives a great degree of freedom in the choice of the phase function, this will be exploited below. In fact if $(x_0, \xi_0) \in \Lambda$ and if $\varphi = \varphi(x, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ parameterizes Λ in a neighborhood U, small enough, of (x_0, ξ_0) then for each $u_h \in I(\mathbb{R}^n, \Lambda)$ and $\chi \in C_0^\infty(T^*\mathbb{R}^n)$, $\operatorname{supp}(\chi) \subset U$, there exists a sequence of amplitudes $a_j = a_j(x, \theta) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ such that for all integer L:

$$\chi^w(x, hD_x)u_h = \sum_{-d \leq j < L} h^j I(a_j e^{\frac{i}{h}\varphi}) + \mathcal{O}(h^L). \tag{9}$$

We will use this remark with the following result of Hörmander (see [11], tome 4, proposition 25.3.3). Let $(T, \tau, x_0, \xi_0, y_0, -\eta_0) \in \Lambda_{\text{flow}}$, $\eta_0 \neq 0$, then near this point there exists, after perhaps a change of local coordinates in y near y_0, a function $S(t, x, \eta)$ such that:

$$\phi(t, x, y, \eta) = S(t, x, \eta) - \langle y, \eta \rangle, \tag{10}$$

parameterizes Λ_{flow}. In particular this implies that:

$$\{(t, \partial_t S(t, x, \eta), x, \partial_x S(t, x, \eta), \partial_\eta S(t, x, \eta), -\eta)\} \subset \Lambda_{\text{flow}},$$

and that the function S is a generating function of the flow, i.e.

$$\Phi_t(\partial_\eta S(t,x,\eta),\eta) = (x, \partial_x S(t,x,\eta)).\tag{11}$$

Moreover, S satisfies the Hamilton-Jacobi equation:

$$\begin{cases} \partial_t S(t,x,\eta) + p_0(x,\partial_x S(t,x,\eta)) = 0, \\ \qquad\qquad S(0,x,\xi) = \langle x,\xi\rangle. \end{cases}$$

Now, we apply this result with $(x_0,\xi_0) = (y_0,\eta_0)$, our unique fixed point of the flow on the energy surface Σ_{E_c}. If $\xi_0 = 0$ we can replace the operator P_h by $e^{\frac{i}{h}\langle x,\xi_1\rangle} P_h e^{-\frac{i}{h}\langle x,\xi_1\rangle}$ with $\xi_1 \neq 0$. This will not change the spectrum since this new operator has the symbol $p(x,\xi-\xi_1,h)$ and the critical point is now (x_0,ξ_1) with $\xi_1 \neq 0$. Consequently, the localized trace $\gamma_2(E_c,h)$, defined by Eq.(6), can be written for all $N \in \mathbb{N}$ and modulo $\mathcal{O}(h^N)$ as:

$$\gamma_2(E_c,h) = \sum_{j<N}(2\pi h)^{-d+j}\int_{\mathbb{R}\times\mathbb{R}^{2n}} e^{\frac{i}{h}(S(t,x,\xi)-\langle x,\xi\rangle+tE_c)} a_j(t,x,\xi)\hat\varphi(t)dtdxd\xi.$$

$$\tag{12}$$

To obtain the right power $-d$ of h occurring in Eq.(12) we apply results of Duistermaat [8] (following here Hörmander for the FIO, see [12] tome 4, for example) concerning the order. An h-pseudodifferential operator obtained by Weyl quantization:

$$(2\pi h)^{-\frac{N}{2}}\int_{\mathbb{R}^N} a(\frac{x+y}{2},\xi)e^{\frac{i}{h}\langle x-y,\xi\rangle}d\xi,$$

is of order 0 w.r.t. $1/h$. Now, since the order of $U_h(t)\Theta(P_h)$ is $-\frac14$, we find that

$$\psi^w(x,hD_x)U_h(t)\Theta(P_h) \sim \sum_{j<N}(2\pi h)^{-n+j}\int_{\mathbb{R}^n} a_j(t,x,y,\eta)e^{\frac{i}{h}(S(t,x,\eta)-\langle y,\eta\rangle)}dy,$$

$$\tag{13}$$

Multiplying Eq.(13) by $\hat\varphi(t)e^{\frac{i}{h}tE_c}$ and passing to the trace we find (12) with $d = n$, where we write again $a_j(t,x,\eta)$ for $a_j(t,x,x,\eta)$.

To each element u_h of $I(\mathbb{R}^n,\Lambda)$ we can associate a principal symbol $e^{\frac{i}{h}S}\sigma_{\mathrm{princ}}(u_h)$, where S is a function on Λ such that $\xi dx = dS$ on Λ. If $u_h = I(ae^{\frac{i}{h}\varphi})$, then we have $S = S_\varphi = \varphi \circ i_\varphi^{-1}$ and $\sigma_{\mathrm{princ}}(u_h)$ is a section of $|\Lambda|^{\frac12} \otimes M(\Lambda)$, where $M(\Lambda)$ is the Maslov vector-bundle of Λ and $|\Lambda|^{\frac12}$ the bundle of half-densities on Λ. If p_1 is the sub-principal symbol of P_h, the half-density of the propagator $U_h(t)$ is easily expressed in the global coordinates (t,y,η) on Λ_{flow} via:

$$\exp(i\int_0^t p_1(\Phi_s(y,-\eta))ds)|dtdyd\eta|^{\frac12}.\tag{14}$$

This expression is related to the resolution of the first transport equation for the propagator. For a proof we refer to Duistermaat and Hörmander [9].

4 Classical dynamics near the critical point.

A critical point of the phase function of Eq.(12) leads to the equations:

$$\left\{ \begin{array}{l} E_c = -\partial_t S(t,x,\xi), \\ x = \partial_\xi S(t,x,\xi), \\ \xi = \partial_x S(t,x,\xi), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} p_0(x,\xi) = E_c, \\ \Phi_t(x,\xi) = (x,\xi), \end{array} \right.$$

where the right hand side defines a closed trajectory of the flow inside Σ_{E_c}. Since we are interested in the contribution of the critical point, we choose a function $\psi \in C_0^\infty(T^*\mathbb{R}^n)$, with $\psi = 1$ near z_0, hence:

$$\begin{aligned} \gamma_2(E_c, h) &=& \frac{1}{2\pi} \mathrm{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t) \psi^w(x, hD_x) \exp(-\frac{i}{h} tP_h) \Theta(P_h) dt \\ &+& \frac{1}{2\pi} \mathrm{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t)(1 - \psi^w(x, hD_x)) \exp(-\frac{i}{h} tP_h) \Theta(P_h) dt. \end{aligned}$$

The contribution on $\mathrm{supp}(1 - \psi)$ is non-singular and can be treated by the regular trace formula. If $\mathrm{supp}(\psi)$ is small enough only points $(t, x, \xi) = (t, z_0)$, for $t \in \mathrm{supp}(\hat{\varphi})$, will give a contribution, since z_0 is isolated on Σ_{E_c}.

Now, we restrict our attention to the contribution of the critical point. Until further notice, the derivatives d will be taken with respect to initial conditions. Since z_0 is totally degenerate, we obtain:

$$d\Phi_t(z_0) = \exp(0) = \mathrm{Id}, \ \forall t. \tag{15}$$

With (H_2), the next homogeneous components of the flow are given by:

$$d^j \Phi_t(z_0) = 0, \ \forall t, \ \forall j \in \{2, .., k-2\}. \tag{16}$$

To obtain the next non-zero terms of the Taylor expansion of the flow, we will use the following technical result:

Lemma 11 *Let be z_0 an equilibrium of the C^∞ vector field X and Φ_t the flow of X. Then for all $m \in \mathbb{N}^*$, there exists a polynomial map P_m, vector valued and of degree at most m, such that:*

$$d^m \Phi_t(z_0)(z^m) = d\Phi_t(z_0) \int_0^t d\Phi_{-s}(z_0) P_m(d\Phi_s(z_0)(z), ..., d^{m-1}\Phi_s(z_0)(z^{m-1})) ds. \tag{17}$$

Proof. We note $x^l \in (\mathbb{R}^n)^l$ the image of x under the diagonal mapping, with the same convention for any vector. If f, g are two C^∞ applications, by the "Faa di Bruno formula", we have:

$$d^m(f o g)(x_0)(x^m) = \sum_{p \in \mathcal{P}(m)} C(p) d^r f(g(x_0))(d^{n_1} g(x_0)(x^{n_1}), ..., d^{n_r} g(x_0)(x^{n_r})),$$

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where $\mathcal{P}(m)$ is the set of partitions of the integer m and where it is assumed that the partition p of m is given by $m = n_1 + ... + n_r$ and $C(\mathrm{p})$ are some universal integers. Hence, for our fixed point z_0 we obtain:

$$d^m(X o \Phi_s)(z_0)(z^m) = \sum_{\mathrm{p} \in \mathcal{P}(m)} C(\mathrm{p}) d^r X(z_0)(d^{n_1} \Phi_s(z_0)(z^{n_1}), ..., d^{n_r} \Phi_s(z_0)(z^{n_r})).$$

For $Y = (Y_1, ..., Y_m)$, we can define:

$$P_m(Y) = \sum_{\mathrm{p} \in \mathcal{P}(m)} C(\mathrm{p}) d^r X(z_0)(Y_{n_1}, ..., Y_{n_r}) - dX(z_0)(Y_m), \qquad (18)$$

this leads to the differential equation, operator valued:

$$\frac{d}{ds}(d^m \Phi_s(z_0))(z^m) = dX(z_0) d^m \Phi_s(z_0)(z^m) + P_m(d\Phi_s(z_0)(z), ..., d^{m-1} \Phi_s(z_0)(z^{m-1})).$$

With the initial condition $d^m \Phi_0(z_0) = 0$, the solution is given by Eq.(17). $\blacksquare$

Since $d\Phi_t(z_0) = \mathrm{Id}, \forall t$, the first non-zero term of the Taylor expansion is:

$$d^{k-1} \Phi_t(z_0)(z^{k-1}) = \int_0^t d^{k-1} H_{p_0}(z_0)(z^{k-1}) ds = t d^{k-1} H_{\mathfrak{p}_k}(z_0)(z^{k-1}), \qquad (19)$$

where the last identity is obtained using that $d^{k-1} H_{p_0}(z_0) = d^{k-1} H_{\mathfrak{p}_k}(z_0)$. Moreover, with $d^2 p_0(z_0) = 0$, for the next term Lemma 11 gives:

$$d^k \Phi_t(z_0)(z^k) = \int_0^t d^k H_{p_0}(z_0)(z^k) ds = t d^k H_{\mathfrak{p}_{k+1}}(z_0)(z^k).$$

Remark 12 For the derivatives $d^j \Phi_t(z_0)$, with $j > k$ there is two different kind of terms, namely:

$$\int_0^t d^j H_{p_0}(z_0)(z^j) ds = t d^j H_{p_0}(z_0)(z^j) = \mathcal{O}(||z||^j),$$

and terms involving powers of t. For example, we have:

$$\int_0^t d^{j+2-k} H_{p_0}(z_0)(z^{j+1-k}, d^{k-1} \Phi_t(z_0)(z^{k-1}))$$

$$= \frac{t^2}{2} d^{j+2-k} H_{p_0}(z_0)(z^{j+1-k}, d^{k-1} H_{\mathfrak{p}_k}(z_0)(z^{k-1})).$$

This term is simultaneously $\mathcal{O}(t^2)$ and $\mathcal{O}(||z||^j)$ near $(0, z_0)$. A similar result holds for other terms, which are $\mathcal{O}(t^d)$ for $d \geq 2$, by an easy recurrence.

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Lemma 13 *Near z_0, here supposed to be 0 to simplify, we have:*

$$S(t,x,\xi) - \langle x,\xi \rangle + tE_c = -t(\mathfrak{p}_k(x,\xi) + R_{k+1}(x,\xi) + tG_{k+1}(t,x,\xi)), \quad (20)$$

where $R_{k+1}(x,\xi) = \mathcal{O}(||(x,\xi)||^{k+1})$ and $G_{k+1}(t,x,\xi) = \mathcal{O}(||(x,\xi)||^{k+1})$, uniformly for t in a compact subset of $\mathbb{R}$.

Proof. By Taylor and under hypothesis (H_2) we obtain:

$$\Phi_t(x,\xi) = (x,\xi) + \frac{1}{(k-1)!}d^{k-1}\Phi_t(0)(z^{k-1}) + \mathcal{O}(||z||^k). \quad (21)$$

Now, we search our local generating function as:

$$S(t,x,\xi) = -tE_c + \langle x,\xi \rangle + \sum_{j=3}^{N} S_j(t,x,\xi) + \mathcal{O}(||(x,\xi)||^{N+1}),$$

where the functions S_j are time dependant and homogeneous of degree j w.r.t. (x,ξ). With the implicit relation: $\Phi_t(\partial_\xi S(t,x,\xi),\xi) = (x,\partial_x S(t,x,\xi))$ and using Eq.(21) we have:

$$S(t,x,\xi) = -tE_c + \langle x,\xi \rangle + S_k(t,x,\xi) + \mathcal{O}(||(x,\xi)||^{k+1}),$$

and comparing terms of degree $k-1$ gives:

$$J\nabla S_k(t,x,\xi) = -\frac{1}{(k-1)!}d^{k-1}\Phi_t(0)((x,\xi)^{k-1}),$$

where J is the matrix of the usual symplectic form. By homogeneity and with Eq.(19) we obtain that:

$$S_k(t,x,\xi) = \frac{1}{k!}\left\langle (x,\xi), tJd^{k-1}H_{\mathfrak{p}_k}(x,\xi)^{k-1} \right\rangle = -t\mathfrak{p}_k(x,\xi).$$

It remains now to treat the remainder. Since $S(0,x,\xi) = \langle x,\xi \rangle$, we have:

$$S(t,x,\xi) - \langle x,\xi \rangle = tF(t,x,\xi),$$

with F smooth in a neighborhood of $(x,\xi) = 0$. Now, the Hamilton-Jacobi equation imposes that $F(0,x,\xi) = -p_0(x,\xi)$ and we obtain:

$$R_{k+1}(x,\xi) = p_0(x,\xi) - E_c - \mathfrak{p}_k(x,\xi) = \mathcal{O}(||(x,\xi)||^{k+1}).$$

Finally, the time dependant remainder can be written:

$$S(t,x,\xi) - S(0,x,\xi) - t\partial_t S(0,x,\xi) = \mathcal{O}(t^2),$$

since by construction this term is of order $\mathcal{O}(||(x,\xi)||^{k+1})$ we get the result. $\blacksquare$

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5 Normal forms of the phase function.

Since the contribution we study is local, we can work with some coordinates and identify locally $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ near the critical point. We define:

$$\Psi(t,z) = \Psi(t,x,\xi) = S(t,x,\xi) - \langle x,\xi\rangle + tE_c, \ z = (x,\xi) \in \mathbb{R}^{2n}. \qquad (22)$$

Lemma 14 *If P_h satisfies conditions (H_2) and (H_4) then, in a neighborhood of $z = z_0$, there exists local coordinates χ such that $\Psi(t,z) \simeq \chi_0\chi_1^k$ if z_0 is a maximum and $\Psi(t,z) \simeq -\chi_0\chi_1^k$ if z_0 is minimum.*

Proof. We can here assume that z_0 is the origin and we use polar coordinates $z = (r,\theta)$, $\theta \in \mathbb{S}^{2n-1}(\mathbb{R})$. With Lemma 13, near the critical point we have:

$$\Psi(t,z) \simeq \Psi(t,r\theta) = -tr^k(\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t,r\theta)),$$

where $\mathfrak{p}_k(\theta)$ is the restriction of $\mathfrak{p}_k$ on $\mathbb{S}^{2n-1}$. We define new coordinates:

$$(\chi_0, \chi_2, ..., \chi_{2n})(t,r,\theta) = (t, \theta_1, ..., \theta_{2n-1}),$$
$$\chi_1(t,r,\theta) = r|\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t,r\theta)|^{\frac{1}{k}}.$$

In these coordinates the phase becomes $-\chi_0\chi_1^k$ for a minimum and $\chi_0\chi_1^k$ for a maximum. Now, we observe that:

$$\frac{\partial\chi_1}{\partial r}(t,0,\theta) = |\mathfrak{p}_k(\theta)|^{\frac{1}{k}}, \ \forall t,$$

hence, the corresponding Jacobian satisfies the relation:

$$|J\chi|(t,0,\theta) = |\mathfrak{p}_k(\theta)|^{\frac{1}{k}} \neq 0, \ \forall t,$$

and this defines a local system of coordinates near z_0 for all t. $\blacksquare$

This change of coordinates leads to:

$$\int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} e^{\frac{i}{\hbar}\Psi(t,r,\theta)}a(t,r\theta)r^{2n-1}dtdrd\theta = \int e^{\pm\frac{i}{\hbar}\chi_0\chi_1^k}A(\chi_0,\chi_1)d\chi_0d\chi_1,$$

with a new amplitude A defined by:

$$A(\chi_0,\chi_1) = \int \chi^*(a(t,r\theta)r^{2n-1}|J\chi|)d\chi_2...d\chi_{2n}. \qquad (23)$$

Remark 15 *Since $\chi_1(t,r,\theta) = r|\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t,r\theta)|^{\frac{1}{k}}$, our new amplitude satisfies $A(\chi_0,\chi_1) = \mathcal{O}(\chi_1^{2n-1})$, near $\chi_1 = 0$. This will play a major role since distributions of Lemma 16 below are supported in $\{\chi_1 = 0\}$.*

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We end this section with a lemma on asymptotics of oscillatory integrals.

Lemma 16 *There exists a sequence $(c_j)_j$ of distributions, whose support is contained in the set $\{\chi_1 = 0\}$, such that for all function $a \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$:*

$$\int_0^\infty (\int_\mathbb{R} e^{i\lambda\chi_0\chi_1^k} a(\chi_0,\chi_1)d\chi_0)d\chi_1 \sim \sum_{j=0}^\infty \lambda^{-\frac{j+1}{k}} c_j(a), \qquad (24)$$

asymptotically for $\lambda \to \infty$, with

$$c_j = \frac{1}{k}\frac{1}{j!}(\mathcal{F}(x_-^{\frac{j+1-k}{k}})(\chi_0) \otimes \delta_0^{(j)}(\chi_1)), \ x_- = \max(-x,0).$$

Proof. We note (t,r) for (χ_0,χ_1) and we define $\hat{g}(\tau,r) = \mathcal{F}_t(a(t,r))(\tau)$, where $\mathcal{F}_t$ is the partial Fourier transform with respect to t. Then, we obtain:

$$\int_0^\infty (\int_\mathbb{R} e^{i\lambda t r^k} a(t,r)dt)dr = \int_0^\infty \hat{g}(-\lambda r^k,r)dr = \lambda^{-\frac{1}{k}} \int_0^\infty \hat{g}(-r^k,\frac{r}{\lambda^{\frac{1}{k}}})dr.$$

A Taylor expansion with respect to r for $\hat{g}(\tau,r)$ at the origin gives:

$$\hat{g}(-r^k,\frac{r}{\lambda^{\frac{1}{k}}}) = \sum_{l=0}^N \frac{\lambda^{-\frac{l}{k}}}{l!} r^l \frac{\partial^l \hat{g}}{\partial r^l}(-r^k,0) + \lambda^{-\frac{N+1}{k}} R_{N+1}(r,\lambda),$$

where $R_{N+1}(r,\lambda)$ is integrable with respect to r, with L^1 norm uniformly bounded in λ. By a new change of variable we obtain:

$$\lambda^{-\frac{1}{k}} \int_0^\infty \hat{g}(-r^k,\frac{r}{\lambda^{\frac{1}{k}}})dr = \frac{1}{k}\sum_{l=0}^N \frac{1}{l!}\lambda^{-\frac{1+l}{k}} \int_{-\infty}^0 \frac{\partial^l \hat{g}}{\partial r^l}(r,0)|r|^{\frac{l+1-k}{k}}dr + \mathcal{O}(\lambda^{-\frac{N+1}{k}}).$$

If we introduce $x_- = \max(-x,0)$ and $\mathcal{F}(x_-^{\frac{l+1-k}{k}})(r)$ the result follows. ∎

Remark 17 A similar result holds for the phase $-\chi_0\chi_1^k$ if we change χ_0 into $-\chi_0$ and we then have simply to replace terms x_- by $x_+ = \max(x,0)$.

6   Proof of the main result.

We can assume that z_0 is a maximum. Lemma 16 shows that the first non-zero coefficient is obtained for $l = 2n - 1$ (see Remark 15) and is given by:

$$\frac{1}{k}\frac{1}{(2n-1)!}\left\langle (\mathcal{F}(x_-^{\frac{2n-k}{k}}) \otimes \delta_0^{(2n-1)}), A(\chi_0,\chi_1)\right\rangle = \frac{1}{k}\int (\mathcal{F}(x_-^{\frac{2n-k}{k}})(\chi_0)\tilde{A}(\chi_0,0)d\chi_0,$$

where, by construction, the new amplitude is:

$$\tilde{A}(\chi_0,\chi_1) = \int \chi^*(a(t,r\theta)|J\chi||\mathfrak{p}_k(\theta)+rR_{k+1}(\theta)+tG_{k+1}(t,r\theta)|^{-\frac{2n-1}{k}})d\chi_2...d\chi_{2n}.$$

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Now, we use an oscillatory representation of $\tilde{A}(\chi_0, 0)$ via:

$$\tilde{A}(\chi_0, 0) = \frac{1}{2\pi} \int \tilde{A}(\chi_0, \chi_1) e^{iz\chi_1} dz d\chi_1.$$

We return to the initial coordinates and, since $\chi_0 = t$, we obtain:

$$\tilde{A}(t, 0) = \frac{1}{2\pi} \int a(t, r\theta) |\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t, r\theta)|^{-\frac{2n-1}{k}} e^{iz\chi_1(t,r,\theta)} dz dr d\theta.$$

Inserting the definition of χ_1, we use the change of variable:

$$y = z|\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t, r\theta)|^{\frac{1}{k}}.$$

Since in $r = 0$ we have $G_{k+1}(t, 0) = 0$, by integration in (y, r) we obtain:

$$\tilde{A}(\chi_0, 0) = \int_{\mathbb{S}^{2n-1}} a(\chi_0, 0) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta. \tag{25}$$

The distribution $\mathcal{F}(x_-^{\frac{2n-k}{k}})(\chi_0)$ of Lemma 16 acts on this function via:

$$\left\langle \mathcal{F}(x_-^{\frac{2n-k}{k}})(\chi_0), \tilde{A}(\chi_0, 0) \right\rangle = \left\langle \mathcal{F}(x_-^{\frac{2n-k}{k}}), a(., 0) \right\rangle \int_{\mathbb{S}^{2n-1}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta.$$

Hence, the top-order contribution to the trace formula is given by:

$$\gamma(E_c, h) = \frac{1}{k} \frac{h^{-n}}{(2\pi)^{n+1}} h^{\frac{2n}{k}} \left\langle \mathcal{F}(x_-^{\frac{2n-k}{k}}), a(., 0) \right\rangle \int_{\mathbb{S}^{2n-1}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta + \mathcal{O}(h^{\frac{2n+1}{k}-n}).$$

With $a(t, 0) = \hat{\varphi}(t) \exp(it p_1(z_0))$, see (14), by Fourier inversion we have:

$$\frac{1}{2\pi} \left\langle \mathcal{F}(x_-^{\frac{2n-k}{k}}), a(., 0) \right\rangle = \int_{\mathbb{R}} \varphi(t + p_1(z_0)) t_-^{\frac{2n-k}{k}} dt.$$

Finally, if z_0 is a local minimum we must simply replace t_- by $t_+ = \max(t, 0)$ and this complete the proof of Theorem 2. ∎

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